Homework 1

February 2025

Instructions

Do these problems soon. For the traintracks one, do your best to come up with an answer by Friday 9/5; the same goes for reading the handout from Spivak and installing Isabelle. For others, aim for Wed 9/10 or sooner.

1 Geometry

1.A Problem

[25 min]

This is the ”traintracks” problem handed out in class.

Here’s highly stylized drawing of train tracks disappearing towards the horizon.

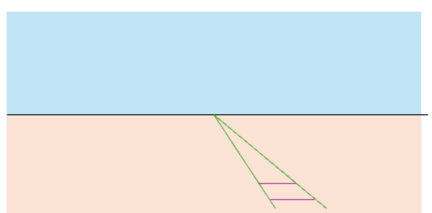

Figure 1: Train tracks disappearing to the horizon, with only two ‘sleepers’ drawn. Show how to draw the next two to keep their separation in proportion.

Because they are train tracks, you know that the distance between them is constant, i.e., they are parallel lines, but in this projection of the world onto a flat surface, they appear to converge to a point on the horizon.

Because they are train tracks, and there are standards for these things, you also know that the "sleepers" or "rail ties" — the pieces of wood that the track sits on — are equally-spaced along the track, and are parallel to each other.

Unfortunately for you, I've only drawn two adjacent sleepers. Your challenge is to draw the rest of them (or at least several more). You *know* that as you move towards the horizon in the picture, the sleepers will get closer and closer together in the picture (even though they're equally spaced in the real world). Your challenge is to draw them getting closer together in a way that's geometrically correct. Frankly, a good drawing of how you produce one or two more sleepers is all that's needed to show that you 'get it.'

You may assume that things that appear parallel in the drawing are in fact intended to be parallel. For instance, the two rail-ties I've drawn are parallel not only in the world depicted, but also in the depiction.

Note: With the correct search terms, you can surely find the exact picture that you need to draw via Google. That will prove that you know how to use a search engine, but not that you have any insight into geometry. When I ask you a followup question in class, you're going to feel kind of silly, right? So don't do that.

Again, submit a picture to GradeScope.

1.B Problem

[20 min] The axioms for an affine plane don't mention measurement anywhere. The axioms for an affine 3-space are similar, except that in addition to points and lines, there are also planes. The 3-space axioms for a space S (I haven't written down all of them) tell you that

- Each plane in S is an affine plane
- Each line of a plane H (considered as an affine plane) is a line of S as well.
- There are four non-coplanar points
- The intersection of two non-parallel planes H_1 and H_2 in S is a line of S, *and* a line in each of the affine planes H_1 and H_2. ("parallel" again means "no shared points, or equal")
- Similarly, the intersection of a line k and a plane H of S is a point of S, and a point of the affine plane H.
- There's a plane through any three noncollinear points of S

From these additional facts you get a few more things: if you have a line k and a point P of S, and P is not on the line k, then there's a plane containing both k and P. (Proof: pick two points of k, together with P, and form the plane H.) If two distinct points of S lie in the plane H, then the line between them is a line of H and also a line of S.

A little thought should convince you that ordinary 3-space is in fact an affine 3-space.

Figure 2 gives a picture of a paralleliped (which I'm going to call a 'box' because it's shorter to type) in 3-space (i.e., a cube-like thing made up from 12 line segments, which are parallel in groups of four, indicated by color – notice that no mention of angle measure or length is included here). From the axioms of an affine-3-space, one can prove some of the things that look apparent from this picture, like "opposite faces of the box lie in parallel planes".[1] Suppose

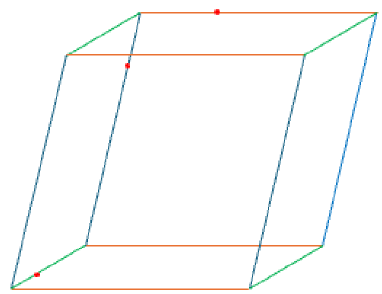

Figure 2: A parallelipiped, with three red dots indicating part of its intersection with a plane.

that I tell you that a particular plane H intersects the box, and the intersection contains the three red dots (although there are obviously other intersections as well). Show how, using affine-geometry facts only, you can draw the rest of the intersection set. Feel free to talk about 'segments' of lines, or alternatively, to discuss all the lines of H that contain a segment of the intersection.

The goal of this problem isn't to get terribly formal, but instead to give you practice thinking about geometric things without using measurements.

Produce a picture somehow – drawing with some drawing tool, writing on paper and taking a picture with your phone – and hand it in to GradeScope.

2 Mathematics

2.A Exercise

[40 min] Read the extract from Spivak's *Calculus* (the page from the Preface and the somewhat larger portion of the Prologue). You'll want to read it carefully, because we'll be 'formalizing' that prologue material in Isabelle. Since you've taken Abstract Algebra, there should be little in the field axioms portion to confuse you. The material about order (positive, negative, less-than, greater-than) may be less familiar, but should not be surprising. You might want, as you

[1]Actually *defining* the box is tricky – you have to talk about subsets of lines, and that requires 'order properties', i.e., a notion of between-ness. We're just going to ignore that for now.

read both parts, to ask yourself ‘Is Spivak skipping steps?’ Hint: The answer is ‘Yes.’[2] Noting such gaps will help prepare you for the formalization process.

The extract is appended to this document.

3 Isabelle

3.A Exercise

[20 min] (This exercise is best done side by side with another classmate or two.) Install Isabelle2025 on your computer, and make sure it works by typing into the main panel of the interface — the one that has ”Scratch.thy” at the top — the following:

```
theory hw1
  imports Main
begin
end
```

and then saving it with the file name “hw1.thy”. Resolve any problems that arise (e.g., “Warning: Isabelle2025 wants to access files on the Desktop”), and share your solutions to these problems via EdStem if they’re not already described there.

3.B Exercise

[3 min] Edit the ‘imports’ line in your hw1.thy file to include `Complex_Main`, a theory that includes the definition of the real and the complex numbers.

3.C Exercise

[5 min] Figure out how to use `find_theorems` in your theory (between the ”begin” and the ”end”) to look for theorems involving the name Cauchy. Here’s a query that looks for theorems whose names include the word `diff`:

```
find_theorems name: "*diff*"
```

If you seem to be finding nothing, you probably need a slightly more complete query. Search for `find_theorems` in the isar-ref manual that’s available in the left column of the Isabelle interface, and try to figure out how to phrase your query. Hint: you really really need to have `Complex_Main` imported into your theory for this to work.

3.D Exercise

[5 min] Comment out that search, but leave it in the document. (This is just to remember how to type inline comments in Isabelle.) Before you do so, paste in a random search-result from your search, and include it in the comment, so we know you succeeded.

[2] Hat tip to Tom Doeppner, who used to ask questions like this on his exams.

3.E Exercise

[5 min] Figure out how to use the *Search Panel* to look for theorems involving the name "Cauchy".

3.F Exercise

[15 min] Type in the following small lemma and deal with any errors. (There will be some!) Be sure to enter the lemma between the 'begin' and 'end' that mark the body of your document.

```
lemma: 2 + 2 = 4
sorry
```

- Give the lemma a name, and fix any errors that arise as you do so. (Treat "and fix errors" as part of all subsequent problems!)
- Replace "sorry" with a proof of the lemma, preferably without using sledgehammer or try or try0.
- Add a second named lemma of your own. (Make it simple, but not identical to the first one), and try to prove it as well.
- Save the file and hand it in to Gradescope

PREFACE TO THE FIRST EDITION

Every aspect of this book was influenced by the desire to present calculus not merely as a prelude to but as the first real encounter with mathematics. Since the foundations of analysis provided the arena in which modern modes of mathematical thinking developed, calculus ought to be the place in which to expect, rather than avoid, the strengthening of insight with logic. In addition to developing the students' intuition about the beautiful concepts of analysis, it is surely equally important to persuade them that precision and rigor are neither deterrents to intuition, nor ends in themselves, but the natural medium in which to formulate and think about mathematical questions.

This goal implies a view of mathematics which, in a sense, the entire book attempts to defend. No matter how well particular topics may be developed, the goals of this book will be realized only if it succeeds as a whole. For this reason, it would be of little value merely to list the topics covered, or to mention pedagogical practices and other innovations. Even the cursory glance customarily bestowed on new calculus texts will probably tell more than any such extended advertisement, and teachers with strong feelings about particular aspects of calculus will know just where to look to see if this book fulfills their requirements.

A few features do require explicit comment, however. Of the twenty-nine chapters in the book, two (starred) chapters are optional, and the three chapters comprising Part V have been included only for the benefit of those students who might want to examine on their own a construction of the real numbers. Moreover, the appendices to Chapters 3 and 11 also contain optional material.

The order of the remaining chapters is intentionally quite inflexible, since the purpose of the book is to present calculus as the evolution of one idea, not as a collection of "topics." Since the most exciting concepts of calculus do not appear until Part III, it should be pointed out that Parts I and II will probably require less time than their length suggests—although the entire book covers a one-year course, the chapters are not meant to be covered at any uniform rate. A rather natural dividing point does occur between Parts II and III, so it is possible to reach differentiation and integration even more quickly by treating Part II very briefly, perhaps returning later for a more detailed treatment. This arrangement corresponds to the traditional organization of most calculus courses, but I feel that it will only diminish the value of the book for students who have seen a small amount of calculus previously, and for bright students with a reasonable background.

The problems have been designed with this particular audience in mind. They range from straightforward, but not overly simple, exercises which develop basic techniques and test understanding of concepts, to problems of considerable difficulty and, I hope, of comparable interest. There are about 625 problems in all. Those which emphasize manipulations usually contain many example, numbered

CHAPTER 1 BASIC PROPERTIES OF NUMBERS

The title of this chapter expresses in a few words the mathematical knowledge required to read this book. In fact, this short chapter is simply an explanation of what is meant by the "basic properties of numbers," all of which—addition and multiplication, subtraction and division, solutions of equations and inequalities, factoring and other algebraic manipulations—are already familiar to us. Nevertheless, this chapter is not a review. Despite the familiarity of the subject, the survey we are about to undertake will probably seem quite novel; it does not aim to present an extended review of old material, but to condense this knowledge into a few simple and obvious properties of numbers. Some may even seem too obvious to mention, but a surprising number of diverse and important facts turn out to be consequences of the ones we shall emphasize.

Of the twelve properties which we shall study in this chapter, the first nine are concerned with the fundamental operations of addition and multiplication. For the moment we consider only addition: this operation is performed on a pair of numbers—the sum $a + b$ exists for any two given numbers a and b (which may possibly be the same number, of course). It might seem reasonable to regard addition as an operation which can be performed on several numbers at once, and consider the sum $a_1 + \cdots + a_n$ of n numbers $a_1, \ldots, a_n$ as a basic concept. It is more convenient, however, to consider addition of pairs of numbers only, and to define other sums in terms of sums of this type. For the sum of three numbers a, b, and c, this may be done in two different ways. One can first add b and c, obtaining $b + c$, and then add a to this number, obtaining $a + (b + c)$; or one can first add a and b, and then add the sum $a + b$ to c, obtaining $(a + b) + c$. Of course, the two compound sums obtained are equal, and this fact is the very first property we shall list:

(P1) If a, b, and c are any numbers, then

$$a + (b + c) = (a + b) + c.$$

The statement of this property clearly renders a separate concept of the sum of three numbers superfluous; we simply agree that $a + b + c$ denotes the number $a + (b + c) = (a + b) + c$. Addition of four numbers requires similar, though slightly more involved, considerations. The symbol $a + b + c + d$ is defined to mean

$$
\begin{aligned}
&(1)\quad ((a + b) + c) + d,\\
\text{or}\quad &(2)\quad (a + (b + c)) + d,\\
\text{or}\quad &(3)\quad a + ((b + c) + d),\\
\text{or}\quad &(4)\quad a + (b + (c + d)),\\
\text{or}\quad &(5)\quad (a + b) + (c + d).
\end{aligned}
$$

This definition is unambiguous since these numbers are all equal. Fortunately, *this* fact need not be listed separately, since it follows from the property P1 already listed. For example, we know from P1 that

$$(a+b)+c=a+(b+c),$$

and it follows immediately that (1) and (2) are equal. The equality of (2) and (3) is a direct consequence of P1, although this may not be apparent at first sight (one must let $b+c$ play the role of b in P1, and d the role of c). The equalities (3) = (4) = (5) are also simple to prove.

It is probably obvious that an appeal to P1 will also suffice to prove the equality of the 14 possible ways of summing five numbers, but it may not be so clear how we can reasonably arrange a proof that this is so without actually listing these 14 sums. Such a procedure is feasible, but would soon cease to be if we considered collections of six, seven, or more numbers; it would be totally inadequate to prove the equality of all possible sums of an arbitrary finite collection of numbers $a_1, \ldots, a_n$. This fact may be taken for granted, but for those who would like to worry about the proof (and it is worth worrying about once) a reasonable approach is outlined in Problem 24. Henceforth, we shall usually make a tacit appeal to the results of this problem and write sums $a_1 + \cdots + a_n$ with a blithe disregard for the arrangement of parentheses.

The number 0 has one property so important that we list it next:

(P2) If a is any number, then

$$a+0=0+a=a.$$

An important role is also played by 0 in the third property of our list:

(P3) For every number a, there is a number $-a$ such that

$$a+(-a)=(-a)+a=0.$$

Property P2 ought to represent a distinguishing characteristic of the number 0, and it is comforting to note that we are already in a position to prove this. Indeed, if a number x satisfies

$$a+x=a$$

for any one number a, then $x=0$ (and consequently this equation also holds for all numbers a). The proof of this assertion involves nothing more than subtracting a from both sides of the equation, in other words, adding $-a$ to both sides; as the following detailed proof shows, all three properties P1–P3 must be used to justify this operation.

$$\begin{array}{ll} \text{If} & a+x=a, \\ \text{then} & (-a)+(a+x)=(-a)+a=0; \\ \text{hence} & ((-a)+a)+x=0; \\ \text{hence} & 0+x=0; \\ \text{hence} & x=0. \end{array}$$

As we have just hinted, it is convenient to regard subtraction as an operation derived from addition: we consider $a - b$ to be an abbreviation for $a + (-b)$. It is then possible to find the solution of certain simple equations by a series of steps (each justified by P1, P2, or P3) similar to the ones just presented for the equation $a + x = a$. For example:

$$\begin{array}{ll} \text{If} & x + 3 = 5, \\ \text{then} & (x + 3) + (-3) = 5 + (-3); \\ \text{hence} & x + (3 + (-3)) = 5 - 3 = 2; \\ \text{hence} & x + 0 = 2; \\ \text{hence} & x = 2. \end{array}$$

Naturally, such elaborate solutions are of interest only until you become convinced that they can always be supplied. In practice, it is usually just a waste of time to solve an equation by indicating so explicitly the reliance on properties P1, P2, and P3 (or any of the further properties we shall list).

Only one other property of addition remains to be listed. When considering the sums of three numbers a, b, and c, only two sums were mentioned: $(a + b) + c$ and $a + (b + c)$. Actually, several other arrangements are obtained if the order of a, b, and c is changed. That these sums are all equal depends on

(P4) If a and b are any numbers, then

$$a + b = b + a.$$

The statement of P4 is meant to emphasize that although the operation of addition of pairs of numbers might conceivably depend on the order of the two numbers, in fact it does not. It is helpful to remember that not all operations are so well behaved. For example, subtraction does not have this property: usually $a - b \neq b - a$. In passing we might ask just when $a - b$ does equal $b - a$, and it is amusing to discover how powerless we are if we rely only on properties P1–P4 to justify our manipulations. Algebra of the most elementary variety shows that $a - b = b - a$ only when $a = b$. Nevertheless, it is impossible to derive this fact from properties P1–P4; it is instructive to examine the elementary algebra carefully and determine which step(s) cannot be justified by P1–P4. We will indeed be able to justify all steps in detail when a few more properties are listed. Oddly enough, however, the crucial property involves multiplication.

The basic properties of multiplication are fortunately so similar to those for addition that little comment will be needed; both the meaning and the consequences should be clear. (As in elementary algebra, the product of a and b will be denoted by $a \cdot b$, or simply ab.)

(P5) If a, b, and c are any numbers, then

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

(P6) If a is any number, then

$$a \cdot 1 = 1 \cdot a = a.$$

Moreover, $1 \neq 0$.

(The assertion that $1 \neq 0$ may seem a strange fact to list, but we have to list it, because there is no way it could possibly be proved on the basis of the other properties listed—these properties would all hold if there were only one number, namely, 0.)

(P7) For every number $a \neq 0$, there is a number a^{-1} such that

$$a \cdot a^{-1} = a^{-1} \cdot a = 1.$$

(P8) If a and b are any numbers, then

$$a \cdot b = b \cdot a.$$

One detail which deserves emphasis is the appearance of the condition $a \neq 0$ in P7. This condition is quite necessary; since $0 \cdot b = 0$ for all numbers b, there is *no* number 0^{-1} satisfying $0 \cdot 0^{-1} = 1$. This restriction has an important consequence for division. Just as subtraction was defined in terms of addition, so division is defined in terms of multiplication: The symbol a/b means $a \cdot b^{-1}$. Since 0^{-1} is meaningless, $a/0$ is also meaningless—division by 0 is *always* undefined.

Property P7 has two important consequences. If $a \cdot b = a \cdot c$, it does not necessarily follow that $b = c$; for if $a = 0$, then both $a \cdot b$ and $a \cdot c$ are 0, no matter what b and c are. However, if $a \neq 0$, then $b = c$; this can be deduced from P7 as follows:

$$\begin{array}{ll} \text{If} & a \cdot b = a \cdot c \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = a^{-1} \cdot (a \cdot c); \\ \text{hence} & (a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c; \\ \text{hence} & 1 \cdot b = 1 \cdot c; \\ \text{hence} & b = c. \end{array}$$

It is also a consequence of P7 that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$. In fact,

$$\begin{array}{ll} \text{if} & a \cdot b = 0 \text{ and } a \neq 0, \\ \text{then} & a^{-1} \cdot (a \cdot b) = 0; \\ \text{hence} & (a^{-1} \cdot a) \cdot b = 0; \\ \text{hence} & 1 \cdot b = 0; \\ \text{hence} & b = 0. \end{array}$$

(It may happen that $a = 0$ and $b = 0$ are both true; this possibility is not excluded when we say "either $a = 0$ or $b = 0$"; in mathematics "or" is always used in the sense of "one or the other, or both.")

This latter consequence of P7 is constantly used in the solution of equations. Suppose, for example, that a number x is known to satisfy

$$(x - 1)(x - 2) = 0.$$

Then it follows that either $x - 1 = 0$ or $x - 2 = 0$; hence $x = 1$ or $x = 2$.

On the basis of the eight properties listed so far it is still possible to prove very little. Listing the next property, which combines the operations of addition and multiplication, will alter this situation drastically.

(P9) If a, b, and c are any numbers, then

$$a \cdot (b+c) = a \cdot b + a \cdot c.$$

(Notice that the equation $(b+c) \cdot a = b \cdot a + c \cdot a$ is also true, by P8.)

As an example of the usefulness of P9 we will now determine just when $a - b = b - a$:

$$\begin{array}{ll}
\text{If} & a - b = b - a, \\
\text{then} & (a-b) + b = (b-a) + b = b + (b-a); \\
\text{hence} & a = b + b - a; \\
\text{hence} & a + a = (b + b - a) + a = b + b. \\
\text{Consequently} & a \cdot (1+1) = b \cdot (1+1), \\
\text{and therefore} & a = b.
\end{array}$$

A second use of P9 is the justification of the assertion $a \cdot 0 = 0$ which we have already made, and even used in a proof on page 6 (can you find where?). This fact was not listed as one of the basic properties, even though no proof was offered when it was first mentioned. With P1–P8 alone a proof was not possible, since the number 0 appears only in P2 and P3, which concern addition, while the assertion in question involves multiplication. With P9 the proof is simple, though perhaps not obvious: We have

$$\begin{aligned} a \cdot 0 + a \cdot 0 &= a \cdot (0+0) \\ &= a \cdot 0; \end{aligned}$$

as we have already noted, this immediately implies (by adding $-(a \cdot 0)$ to both sides) that $a \cdot 0 = 0$.

A series of further consequences of P9 may help explain the somewhat mysterious rule that the product of two negative numbers is positive. To begin with, we will establish the more easily acceptable assertion that $(-a) \cdot b = -(a \cdot b)$. To prove this, note that

$$\begin{aligned} (-a) \cdot b + a \cdot b &= [(-a) + a] \cdot b \\ &= 0 \cdot b \\ &= 0. \end{aligned}$$

It follows immediately (by adding $-(a \cdot b)$ to both sides) that $(-a) \cdot b = -(a \cdot b)$. Now note that

$$\begin{aligned} (-a) \cdot (-b) + [-(a \cdot b)] &= (-a) \cdot (-b) + (-a) \cdot b \\ &= (-a) \cdot [(-b) + b] \\ &= (-a) \cdot 0 \\ &= 0. \end{aligned}$$

Consequently, adding $(a \cdot b)$ to both sides, we obtain

$$(-a) \cdot (-b) = a \cdot b.$$

The fact that the product of two negative numbers is positive is thus a consequence of P1–P9. In other words, *if we want P1 to P9 to be true, the rule for the product of two negative numbers is forced upon us.*

The various consequences of P9 examined so far, although interesting and important, do not really indicate the significance of P9; after all, we could have listed each of these properties separately. Actually, P9 is the justification for almost all algebraic manipulations. For example, although we have shown how to solve the equation

$$(x-1)(x-2) = 0,$$

we can hardly expect to be presented with an equation in this form. We are more likely to be confronted with the equation

$$x^2 - 3x + 2 = 0.$$

The "factorization" $x^2 - 3x + 2 = (x-1)(x-2)$ is really a triple use of P9:

$$\begin{aligned}(x-1)\cdot(x-2) &= x\cdot(x-2) + (-1)\cdot(x-2)\\ &= x\cdot x + x\cdot(-2) + (-1)\cdot x + (-1)\cdot(-2)\\ &= x^2 + x[(-2)+(-1)] + 2\\ &= x^2 - 3x + 2.\end{aligned}$$

A final illustration of the importance of P9 is the fact that this property is actually used every time one multiplies arabic numerals. For example, the calculation

$$\begin{array}{r} 13\\ \times 24\\ \hline 52\\ 26\\ \hline 312 \end{array}$$

is a concise arrangement for the following equations:

$$\begin{aligned}13\cdot 24 &= 13\cdot(2\cdot 10 + 4)\\ &= 13\cdot 2\cdot 10 + 13\cdot 4\\ &= 26\cdot 10 + 52.\end{aligned}$$

(Note that moving 26 to the left in the above calculation is the same as writing $26 \cdot 10$.) The multiplication $13 \cdot 4 = 52$ uses P9 also:

$$\begin{aligned}13\cdot 4 &= (1\cdot 10 + 3)\cdot 4\\ &= 1\cdot 10\cdot 4 + 3\cdot 4\\ &= 4\cdot 10 + 12\\ &= 4\cdot 10 + 1\cdot 10 + 2\\ &= (4+1)\cdot 10 + 2\\ &= 5\cdot 10 + 2\\ &= 52.\end{aligned}$$

The properties P1–P9 have descriptive names which are not essential to remember, but which are often convenient for reference. We will take this opportunity to list properties P1–P9 together and indicate the names by which they are commonly designated.

(P1)	(Associative law for addition)	$a + (b + c) = (a + b) + c$.
(P2)	(Existence of an additive identity)	$a + 0 = 0 + a = a$.
(P3)	(Existence of additive inverses)	$a + (-a) = (-a) + a = 0$.
(P4)	(Commutative law for addition)	$a + b = b + a$.
(P5)	(Associative law for multiplication)	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(P6)	(Existence of a multiplicative identity)	$a \cdot 1 = 1 \cdot a = a; \quad 1 \neq 0$.
(P7)	(Existence of multiplicative inverses)	$a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$.
(P8)	(Commutative law for multiplication)	$a \cdot b = b \cdot a$.
(P9)	(Distributive law)	$a \cdot (b + c) = a \cdot b + a \cdot c$.

The three basic properties of numbers which remain to be listed are concerned with inequalities. Although inequalities occur rarely in elementary mathematics, they play a prominent role in calculus. The two notions of inequality, $a < b$ (a is less than b) and $a > b$ (a is greater than b), are intimately related: $a < b$ means the same as $b > a$ (thus $1 < 3$ and $3 > 1$ are merely two ways of writing the same assertion). The numbers a satisfying $a > 0$ are called **positive**, while those numbers a satisfying $a < 0$ are called **negative**. While positivity can thus be defined in terms of $<$, it is possible to reverse the procedure: $a < b$ can be defined to mean that $b - a$ is positive. In fact, it is convenient to consider the collection of all positive numbers, denoted by P, as the basic concept, and state all properties in terms of P:

(P10) (Trichotomy law) For every number a, one and only one of the following holds:

- (i) $a = 0$,
- (ii) a is in the collection P,
- (iii) $-a$ is in the collection P.

(P11) (Closure under addition) If a and b are in P, then $a + b$ is in P.

(P12) (Closure under multiplication) If a and b are in P, then $a \cdot b$ is in P.

These three properties should be complemented with the following definitions:

$$\begin{aligned} a > b &\quad \text{if} \quad a - b \text{ is in } P; \\ a < b &\quad \text{if} \quad b > a; \\ a \geq b &\quad \text{if} \quad a > b \text{ or } a = b; \\ a \leq b &\quad \text{if} \quad a < b \text{ or } a = b.^* \end{aligned}$$

Note, in particular, that $a > 0$ if and only if a is in P.

All the familiar facts about inequalities, however elementary they may seem, are consequences of P10–P12. For example, if a and b are any two numbers, then precisely one of the following holds:

(i) $a - b = 0$,
(ii) $a - b$ is in the collection P,
(iii) $-(a - b) = b - a$ is in the collection P.

Using the definitions just made, it follows that precisely one of the following holds:

(i) $a = b$,
(ii) $a > b$,
(iii) $b > a$.

A slightly more interesting fact results from the following manipulations. If $a < b$, so that $b - a$ is in P, then surely $(b + c) - (a + c)$ is in P; thus, if $a < b$, then $a + c < b + c$. Similarly, suppose $a < b$ and $b < c$. Then

$$\begin{aligned} & b - a \text{ is in } P, \\ \text{and} \quad & c - b \text{ is in } P, \\ \text{so} \quad & c - a = (c - b) + (b - a) \text{ is in } P. \end{aligned}$$

This shows that if $a < b$ and $b < c$, then $a < c$. (The two inequalities $a < b$ and $b < c$ are usually written in the abbreviated form $a < b < c$, which has the third inequality $a < c$ almost built in.)

The following assertion is somewhat less obvious: If $a < 0$ and $b < 0$, then $ab > 0$. The only difficulty presented by the proof is the unraveling of definitions. The symbol $a < 0$ means, by definition, $0 > a$, which means $0 - a = -a$ is in P. Similarly $-b$ is in P, and consequently, by P12, $(-a)(-b) = ab$ is in P. Thus $ab > 0$.

The fact that $ab > 0$ if $a > 0$, $b > 0$ and also if $a < 0$, $b < 0$ has one special consequence: $a^2 > 0$ if $a \neq 0$. Thus squares of nonzero numbers are always positive, and in particular we have proved a result which might have seemed sufficiently elementary to be included in our list of properties: $1 > 0$ (since $1 = 1^2$).

*There is one slightly perplexing feature of the symbols $\geq$ and $\leq$. The statements

$$\begin{aligned} 1 + 1 \leq 3 \\ 1 + 1 \leq 2 \end{aligned}$$

are both true, even though we know that $\leq$ could be replaced by $<$ in the first, and by $=$ in the second. This sort of thing is bound to occur when $\leq$ is used with specific numbers; the usefulness of the symbol is revealed by a statement like Theorem 1—here equality holds for some values of a and b, while inequality holds for other values.

The fact that $-a > 0$ if $a < 0$ is the basis of a concept which will play an extremely important role in this book. For any number a, we define the **absolute value** $|a|$ of a as follows:

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0. \end{cases}$$

Note that $|a|$ is always positive, except when $a = 0$. For example, we have $|-3| = 3$, $|7| = 7$, $|1 + \sqrt{2} - \sqrt{3}| = 1 + \sqrt{2} - \sqrt{3}$, and $|1 + \sqrt{2} - \sqrt{10}| = \sqrt{10} - \sqrt{2} - 1$. In general, the most straightforward approach to any problem involving absolute values requires treating several cases separately, since absolute values are defined by cases to begin with. This approach may be used to prove the following very important fact about absolute values.

THEOREM 1 For all numbers a and b, we have

$$|a + b| \leq |a| + |b|.$$

PROOF We will consider 4 cases:

$$\begin{array}{lll} (1) & a \geq 0, & b \geq 0; \\ (2) & a \geq 0, & b \leq 0; \\ (3) & a \leq 0, & b \geq 0; \\ (4) & a \leq 0, & b \leq 0. \end{array}$$

In case (1) we also have $a + b \geq 0$, and the theorem is obvious; in fact,

$$|a + b| = a + b = |a| + |b|,$$

so that in this case equality holds.

In case (4) we have $a + b \leq 0$, and again equality holds:

$$|a + b| = -(a + b) = -a + (-b) = |a| + |b|.$$

In case (2), when $a \geq 0$ and $b \leq 0$, we must prove that

$$|a + b| \leq a - b.$$

This case may therefore be divided into two subcases. If $a + b \geq 0$, then we must prove that

$$\begin{aligned} a + b &\leq a - b, \\ \text{i.e.,} \qquad b &\leq -b, \end{aligned}$$

which is certainly true since $b \leq 0$ and hence $-b \geq 0$. On the other hand, if $a + b \leq 0$, we must prove that

$$\begin{aligned} -a - b &\leq a - b, \\ \text{i.e.,} \qquad -a &\leq a, \end{aligned}$$

which is certainly true since $a \geq 0$ and hence $-a \leq 0$.

Finally, note that case (3) may be disposed of with no additional work, by applying case (2) with a and b interchanged. ▮

Although this method of treating absolute values (separate consideration of various cases) is sometimes the only approach available, there are often simpler methods which may be used. In fact, it is possible to give a much shorter proof of Theorem 1; this proof is motivated by the observation that

$$|a| = \sqrt{a^2}.$$

(Here, and throughout the book, $\sqrt{x}$ denotes the *positive* square root of x; this symbol is defined only when $x \geq 0$.) We may now observe that

$$\begin{aligned}(|a+b|)^2 = (a+b)^2 &= a^2 + 2ab + b^2 \\ &\leq a^2 + 2|a| \cdot |b| + b^2 \\ &= |a|^2 + 2|a| \cdot |b| + |b|^2 \\ &= (|a| + |b|)^2.\end{aligned}$$

From this we can conclude that $|a+b| \leq |a| + |b|$ because $x^2 < y^2$ implies $x < y$, provided that x and y are both nonnegative; a proof of *this* fact is left to the reader (Problem 5).

One final observation may be made about the theorem we have just proved: a close examination of either proof offered shows that

$$|a+b| = |a| + |b|$$

if a and b have the same sign (i.e., are both positive or both negative), or if one of the two is 0, while

$$|a+b| < |a| + |b|$$

if a and b are of opposite signs.

We will conclude this chapter with a subtle point, neglected until now, whose inclusion is required in a conscientious survey of the properties of numbers. After stating property P9, we proved that $a - b = b - a$ implies $a = b$. The proof began by establishing that

$$a \cdot (1+1) = b \cdot (1+1),$$

from which we concluded that $a = b$. This result is obtained from the equation $a \cdot (1+1) = b \cdot (1+1)$ by dividing both sides by $1+1$. Division by 0 should be avoided scrupulously, and it must therefore be admitted that the validity of the argument depends on knowing that $1+1 \neq 0$. Problem 25 is designed to convince you that this fact cannot possibly be proved from properties P1–P9 alone! Once P10, P11, and P12 are available, however, the proof is very simple: We have already seen that $1 > 0$; it follows that $1+1 > 0$, and in particular $1+1 \neq 0$.

This last demonstration has perhaps only strengthened your feeling that it is absurd to bother proving such obvious facts, but an honest assessment of our present situation will help justify serious consideration of such details. In this chapter we have assumed that numbers are familiar objects, and that P1–P12 are merely explicit statements of obvious, well-known properties of numbers. It would be difficult, however, to justify this assumption. Although one learns how to "work with" numbers in school, just what numbers *are*, remains rather vague. A great deal of this book is devoted to elucidating the concept of numbers, and by the end